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## $^3\text{H}$ at Next-to-Next-to-Next-to Leading Order of the Chiral Expansion

Received: 28 September 2012 / Accepted: 6 January 2013 / Published online: 25 January 2013  
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**Abstract** The chiral three-nucleon force (3NF) at next-to-next-to-next-to leading order ( $\text{N}^3\text{LO}$ ) is used to calculate the triton wave function and the doublet nucleon–deuteron scattering length. This allows us to fix the values of the low-energy constants which are free parameters of the theory. The obtained values of these parameters, the expectation values of the kinetic energy, two- and three-body potentials and individual contributions of different parts of 3NF are given.

### 1 Introduction

The structure of the three-nucleon (3N) force is one of the central questions of modern few-nucleon physics. Comparison of the theoretical predictions with data clearly shows the importance of 3NFs. Inclusion of semi-phenomenological models of the 3NF into calculations improves significantly the description of the data. However, in some cases serious discrepancies remain [1–4]. One of the reasons is a lack of consistency between two- and three-body phenomenological potentials. This problem is absent for the nuclear forces derived in the framework of chiral effective field theory [5], where 3NF occurs for the first time at next-to-next-to leading order ( $\text{N}^2\text{LO}$ ) of the chiral expansion. This force was used in few-body calculations and the quality of the description of the data was as good as with the so-called realistic nuclear forces [6, 7]. That calls for the inclusion of higher order terms of the chiral NN and 3N interactions. Recently an important step along this line was made and the 3NF at  $\text{N}^3\text{LO}$  was derived [8, 9].

The chiral 3NF at  $\text{N}^3\text{LO}$  is much more complex than the one at  $\text{N}^2\text{LO}$ . Besides the  $2\pi$ -exchange force ( $V_{\pi\pi}$ ), its long-range part contains two new topologies. The first one arises from the  $2\pi$  exchange between two nucleons with simultaneous  $1\pi$  exchange with the third nucleon and leads to the  $V_{2\pi-1\pi}$  potential. The second one results from so called ring diagrams where pions are exchanged between all three pairs of nucleons. We

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On 1 August 2012 Walter Glöckle passed away. We dedicate this paper to Walter, who was a great friend and collaborator. Presented at the 20th International IUPAP Conference on Few-Body Problems in Physics, 20–25 August, 2012, Fukuoka, Japan.

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**Table 1** Values of  $c_D$  and  $c_E$  parameters for different combinations of 3NF topologies

No.	3N operators	$c_D$	$c_E$
1	$V_{\pi\pi} + V_{2\pi-1\pi} + V_{ring} + V_{D-term} + V_{E-term}$	11.400	0.560
2	$V_{\pi\pi} + V_{2\pi-1\pi} + V_{ring} + V_{D-term} + V_{E-term} + V_{2\pi-cont}$	13.442	0.206
3	$V_{\pi\pi} + V_{2\pi-1\pi} + V_{ring} + V_{D-term} + V_{E-term} + V_{2\pi-cont} + V_{1/m}$	13.780	0.372

denote this part as  $V_{ring}$ . Also for the short-range part a new topology with contact interaction and  $2\pi$  exchange is present at  $N^3LO$  which leads to the  $V_{2\pi-cont}$  potential. The  $1\pi$ -contact and 3N contact terms which depend on the unknown constants  $c_D$  and  $c_E$ , respectively, have no new contribution at  $N^3LO$ . Thus their operator structure is the same as at  $N^3LO$  leading to  $V_{D-term}$  and  $V_{E-term}$  potentials. Finally, at  $N^3LO$ , first relativistic corrections to the leading  $2\pi$  and  $1\pi$ -contact terms are present and we denote the corresponding potential as  $V_{1/m}$ . For detailed discussion of the chiral 3NF at  $N^3LO$  see [8,9].

## 2 Calculations of Matrix Elements of 3NF at $N^3LO$

The 3N systems which are the lightest systems where the new 3NF can be tested are currently well under theoretical control. Different approaches to 3N systems exist that allow one to obtain reliable results for bound state and scattering observables based on realistic nuclear NN and 3N forces. In most cases a partial wave decomposition (PWD) is used to solve the few-body equations. However, the rich structure of 3NF at  $N^3LO$  makes the traditional PWD very time consuming and impractical. To avoid this obstacle we use a recently developed method [10,11], which utilizes the advantages of software for symbolic algebra. This method allows one to calculate efficiently any 3NF matrix elements in momentum space and was already successfully applied to the Tucson–Melbourne 3NF [11] and some terms of the chiral 3NF [12]. In this method, the PWD of any operator which contributes to the 3NF is reduced to five dimensional numerical integrations over angular variables. The integrand is build from Clebsch–Gordan coefficients, spherical harmonics, isospin factors and matrix elements of the 3NF in spin space. The latter ones are calculated using software for symbolic algebra. The resulting function of Jacobi momenta can be directly used as input for integration. Note that such five dimensional integration has to be repeated for each combination of magnitudes of Jacobi momenta and quantum numbers describing the 3N system.

## 3 Results

### 3.1 Values of Low Energy Constants

The chiral 3NF at  $N^3LO$  depends on two low energy constants, whose values have to be determined from data. To this aim we choose the triton binding energy  $E^3H$  and the neutron–deuteron doublet scattering length  $^2a_{nd}$ . First, for given  $c_D$  and  $c_E$ , we solve the Schrödinger equation for the triton using the method of [13]. Next we solve the Faddeev equation for the 3N system and calculate the transition amplitude for elastic nucleon–deuteron scattering [1], which allows us to obtain the value of  $^2a_{nd}$ . Changing the values of  $c_D$  and  $c_E$  and repeating calculations, we finally obtain the  $c_D$  and  $c_E$  values, which describe the  $E^3H$  and  $^2a_{nd}$  simultaneously. These values of  $c_D$  and  $c_E$  are given in the Table 1 for the chiral force with regularization parameters chosen as  $\Lambda = 450$  MeV and  $\tilde{\Lambda} = 500$  MeV. The  $\Lambda$  parameter is used in regularization of the NN and 3N forces and  $\tilde{\Lambda}$  occurs in the spectral function regularization of the  $2\pi$  exchange NN potential [5]. In addition to the values of  $c_D$  and  $c_E$  obtained separately with the long-range and the leading short-range parts of 3NF given in the first row of Table 1, in the next rows we show  $c_D$  and  $c_E$  resulting from inclusion of short-range  $V_{2\pi-cont}$  (2nd row) and leading relativistic corrections  $V_{1/m}$  (3rd row).

The short-range force  $V_{2\pi-cont}$  changes the  $c_D$  by about 20 % and reduces  $c_E$  by the factor of 2. Next, the relativistic corrections lead to the 80 % correction of  $c_E$ , while  $c_D$  remains nearly unchanged. Note that the values of  $c_D$  and  $c_E$  from the first row of Table 1 differ slightly from the values given in Ref. [12] due to corrected formfactor functions in  $V_{ring}$ .

**Table 2** The expectation values  $\langle H_0 \rangle$ ,  $\langle V_{NN} \rangle$  and  $\langle V_{3N} \rangle$  (in MeV) for the same parametrization of chiral  $\text{N}^3\text{LO}$  potential as given in Table 1

No.	$\langle H_0 \rangle$	$\langle V_{NN} \rangle$	$\langle V_{3N} \rangle$
1	35.963	-43.449	-0.996
2	35.942	-43.399	-1.024
3	35.918	-43.382	-1.017

**Table 3** The expectation values (in MeV) of individual components of the 3NF for the same parametrization of chiral potential at  $\text{N}^3\text{LO}$  as given in Table 1

No.	$\langle V_{\pi\pi} \rangle$	$\langle V_{2\pi-1\pi} \rangle$	$\langle V_{ring} \rangle$	$\langle V_{D-term} \rangle$	$\langle V_{E-term} \rangle$	$\langle V_{2\pi-cont} \rangle$	$\langle V_{1/m} \rangle$
1	-0.648	0.470	0.015	-0.746	-0.087	-	-
2	-0.661	0.485	0.014	-0.912	-0.032	0.082	-
3	-0.655	0.481	0.014	-0.930	-0.057	0.082	0.048

### 3.2 Expectation Values in ${}^3\text{H}$

It is instructive to study expectation values of individual parts of the Hamiltonian although such values are not observable. Table 2 shows such expectation values for the kinetic energy  $\langle H_0 \rangle$ , the NN potential  $\langle V_{NN} \rangle$  and the 3NF  $\langle V_{3N} \rangle$  for the three cases of Table 1. The similarity of the expectation values indicates that there are no strong changes of the wave functions for the three cases.

The expectation values of the individual contributions to the 3NF are shown in Table 3. They do not vary much when new interactions are added. For this parametrization the dominant (negative) contribution comes from  $V_{\pi\pi}$  and  $V_{D-term}$ . However, the magnitude of the latter one depends on the value of  $c_D$ , which is relatively big in the case of this parametrization. The positive contribution of  $V_{2\pi-1\pi}$  is also significant and reaches about 70 % of the  $V_{\pi\pi}$  contribution. The contributions of  $V_{E-term}$  and  $V_{2\pi-cont}$  as well as  $V_{1/m}$  are of the same order: about 10 % of the  $V_{\pi\pi}$  expectation value. Finally, the  $V_{ring}$  expectation value is the smallest and amounts  $\approx 2$  % of the  $V_{\pi\pi}$  contribution. Clearly, calculations of the 3N scattering observables are crucial. First of such calculations [14], which take into account the long-range 3NF supplemented by  $V_{D-term}$  and  $V_{E-term}$  show the dominant role of  $V_{\pi\pi}$  and contact terms, while  $V_{2\pi-1\pi}$  and  $V_{ring}$  are of less importance. This indicates that the sizable  $V_{2\pi-1\pi}$  contributions are effectively canceled, e.g., by the large  $V_{D-term}$ .

**Acknowledgments** This work was supported by the Polish National Science Center under Grant No. DEC-2011/01/B/ST2/00578. It was also partially supported by the European Community-Research Infrastructure Integrating Activity ‘‘Exciting Physics Of Strong Interactions’’ (acronym WP4 EPOS) under the Seventh Framework Programme of the EU and by the European Research Council (ERC-2010-StG 259218 NuclearEFT). The numerical calculations have been performed on the supercomputer clusters of the JSC, Jülich, Germany, and the Ohio Supercomputer Centre, USA (Project PAS0680).

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